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Solitary waves in active-dissipative dispersive media

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Abstract. Solitary waves in active-dissipative dispersive media are considered. The exact solutions in the form of solitary waves and kink-shaped waves are presented. The difference equation for numerical simulation of nonlinear waves is given. Numerical results of the interaction of solitary waves are discussed. It is shown that there is a solitary wave in active-dissipative dispersive media that has the soliton property.

1. Introduction

In this paper we consider solitary waves described by the nonlinear evolution equation in the form

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0. \tag{1.1}$$

Equation (1.1) can be used to describe long waves on a viscous fluid flowing down along an inclined plane [1], unstable drift waves in plasma [2] and stress waves in fragmentated porous media [3].

At $\beta = 0$ equation (1.1) is referred to as the Kuramoto–Sivashinsky equation, one of the simplest equations that appears in modelling the nonlinear behaviour of disturbances for a sufficiently large class of active dissipative media. It represents the evolution of concentration in chemical reactions [4], hydrodynamic instabilities in laminar flame fronts and at the interface of two viscous fluids [5].

The investigation of stability of the steady-state motion described by equation (1.1) with respect to small perturbations $u' \cong \exp\{\mathrm{i}kx + \chi t\}$ gives the linear dispersion relation

$$\chi = k^2(\alpha - \gamma k^2) - \mathrm{i}\beta k^3. \tag{1.2}$$

Equation (1.2) shows that for positive constants α and γ in equation (1.1) small-amplitude sinusoidal waves are linearly unstable for long wavelengths ($0 < k < \sqrt{\alpha/\gamma}$) and stable for short wavelengths ($k > \sqrt{\alpha/\gamma}$). The maximum growth rate occurs at wavenumber $k^* = (\alpha/2\gamma)^{1/2}$.

We multiply equation (1.1) by $u(x,t)$ assuming $\lim_{x\to\pm\infty} u_{mx}(x,t) = 0$ ($m = 0, 1, 2, 3$; u_{mx} is the mth-order derivative with respect to x) and we integrate the obtained expression with respect to x from $-\infty$ to $+\infty$:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle u^2\rangle = d\langle u_x^2\rangle - \gamma\langle u_{xx}^2\rangle \qquad \langle u_x^2\rangle = \int_{-\infty}^{\infty} u_x^2\,\mathrm{d}x. \tag{1.3}$$

One can see from equation (1.3) that, when $\alpha > 0$ and $\gamma > 0$, the term with the second derivative in equation (1.1) corresponds to the addition of energy to the system and the fourth-order derivative term characterizes its dissipation.

The existence of instability and dispersion in equation (1.1) leads to the steady state, because the energy influx due to self-excitation is transferred to short wavelengths and balanced by dumping due to the fourth-derivative dissipation term [6].

Equation (1.1) is not integrable by the inverse scattering transform because this equation does not possess the Painlevé property. The proof of non-integrability of equation (1.1) at $\beta = 0$ is given in [7]. However, equation (1.1) has some special solutions [7–9]. Equation (1.1) is invariant under the Gallilean transformation

$$(u, x, t) \to (u + C, x - Ct, t) \tag{1.4}$$

and we will keep this in mind.

Let us normalize equation (1.1) setting

$$u = \alpha\sqrt{\alpha/\gamma}u' \qquad x = \sqrt{\gamma/\alpha}x' \qquad t = (\gamma/\alpha^2)t' \qquad \sigma = \beta/\sqrt{\alpha\gamma}. \tag{1.5}$$

Then equation (1.1) takes the form

$$u_t + uu_x + u_{xx} + \sigma u_{xxx} + u_{xxxx} = 0 \tag{1.6}$$

(the primes of the variables are omitted). Also, equation (1.6) is invariant under transformations

$$u \to -u \qquad x \to -x \qquad \sigma \to -\sigma \tag{1.7}$$

which allows us to study equation (1.6) for $\sigma \geqslant 0$ only.

The aim of this work is to search for analytical solutions of equation (1.1) by the singular manifold method and analyse them by means of a numerical solution.

2. Special solutions of equation (1.1)

It is known that the Painlevé test provides useful information for identifying the completely integrable cases of families of nonlinear ordinary and partial differential equations [10]. However, the benefit of the Painlevé approach is not limited to the integrability prediction for the integrable equations. Painlevé analysis also yields a systematic procedure for obtaining special solutions when an equation possesses only the conditional Painlevé property [7–9].

As mentioned above, equation (1.1) does not have the Painlevé property but has the conditional Painlevé property. There are a number of approaches to take this into account [7–9], but here we will use the singular manifold method following Conte and Musette [7].

Let us look for special solutions of equation (1.1) in the form

$$u(x, t) = a_0Y^p + a_1Y^{p-1} + \cdots + a_p \tag{2.1}$$

where $Y(x, t)$ satisfies the following set of equations [11]:

$$Y_x = -Y^2 - \tfrac{1}{2}S \tag{2.2}$$

$$Y_t = CY^2 - C_xY + \tfrac{1}{2}(SC + C_{xx}). \tag{2.3}$$

C and S are functions invariant under the Möbius group of transformations [12]. They meet the compatibility condition in the form [7]

$$S_t + C_{xxx} + 2C_xS + CS_x = 0. \tag{2.4}$$

Coefficients $a_0, a_1, \ldots, a_p$ are found after substitution of equation (2.1) into the original equation and equating expressions at different powers of Y. Some of these coefficients can depend on functions S, C and their derivatives.

Inserting $u \cong a_0 Y^p$ into equation (1.6) and taking into account equation (2.2), one can obtain $p = 3$ and $a_0 = 120$. Therefore, the solution of equation (1.6) can be sought in the form

$$u = 120Y^3 + a_1 Y^2 + a_2 Y + a_3. \tag{2.5}$$

Substitution of expression (2.5) into equation (1.6) leads to the following equalities:

$$a_1 = -15\sigma \tag{2.6}$$

$$a_2 = 60S + \frac{15}{76}(16 - \sigma^2) \tag{2.7}$$

$$a_3 = -15S_x - 5\sigma S + C + \frac{\sigma}{76}\left(7 - \frac{13}{8}\sigma^2\right). \tag{2.8}$$

Also, we have the following overdetermined set of equations for S and C with σ as a parameter:

$$-C_x + 3S_{xx} + \frac{5}{4}\sigma S_x + 2S^2 + \frac{5}{152}(16 - \sigma^2)S - \frac{1}{722}\left(\frac{131}{64}\sigma^4 - \frac{87}{8}\sigma^2 + 11\right) = 0 \tag{2.9}$$

$$S_{xxx} + \frac{3}{8}\sigma S_{xx} + 3SS_x + \frac{5}{608}(16 - \sigma^2)S_x + \frac{\sigma}{4}S^2 - \sigma\left[\frac{3}{152}\left(\sigma^2 - \frac{10}{3}\right)\right]^2 = 0 \tag{2.10}$$

$$S_t + S_{xxxx} + \frac{\sigma}{2}S_{xxx} - 2SS_{xx} + \frac{5}{304}(16 - \sigma^2)S_{xx} - 2S_x^2 + (C - \sigma S)S_x + 2SC_x = 0 \tag{2.11}$$

$$\begin{aligned} &C_t + C_x C - 135S_{xx}S_x + \frac{15}{2}\sigma S_{xx}S + \frac{\sigma}{152}\left(\frac{1011\sigma^2}{64} - 39\right)S_{xx} - \frac{225}{8}\sigma S_x^2 - 15S_x S^2 \\ &\quad + \frac{15}{38}\left(\frac{165}{32}\sigma^2 - 1\right)S_x S + 15C_x S_x + \frac{15}{1444}\left(\frac{1991}{2048}\sigma^4 - \frac{581}{128}\sigma^2 + 17\right)S_x \\ &\quad + \frac{45}{4}\sigma S^3 + \frac{3\sigma}{152}\left(\frac{89}{32}\sigma^2 + 3\right)S^2 - 5\sigma\left[\frac{9}{152}\left(\sigma^2 - \frac{10}{3}\right)\right]^2 S \\ &\quad + \frac{\sigma}{152^3}\left(\frac{257}{2}\sigma^6 - \frac{16\,003}{2}\sigma^4 + \frac{97\,037}{2}\sigma^2 - 77\,328\right) = 0. \end{aligned} \tag{2.12}$$

Equations (2.2) and (2.3) can be presented in the linear form [7]

$$\psi_{xx} + \tfrac{1}{2}S\psi = 0 \tag{2.13}$$

$$\psi_t + C\psi_x - \tfrac{1}{2}C_x\psi = 0. \tag{2.14}$$

These equations can be derived after substitution of $Y = \psi_x/\psi$ into equations (2.2) and (2.3).

Taking S and C to be constant is sufficient to lead to solitary wave solutions of equation (1.1). In this case equation (2.4) is evidently satisfied. Also, equations (2.9)–(2.12) are followed by the algebraic equations for S, C and σ as a parameter:

$$2S^2 + \frac{5}{152}(16 - \sigma^2)S - \frac{1}{722}\left(\frac{131}{64}\sigma^4 - \frac{87}{8}\sigma^2 + 11\right) = 0 \tag{2.15}$$

$$\frac{\sigma}{4}S^2 - \sigma\left[\frac{3}{152}\left(\sigma^2 - \frac{10}{3}\right)\right]^2 = 0 \tag{2.16}$$

$$\begin{aligned} &\frac{45}{4}\sigma S^3 + \frac{3\sigma}{152}\left(\frac{89}{32}\sigma^2 + 3\right)S^2 - 5\sigma\left[\frac{9}{152}\left(\sigma^2 - \frac{10}{3}\right)\right]^2 S \\ &\quad + \frac{\sigma}{152^3}\left(\frac{257}{2}\sigma^6 - \frac{16\,003}{2}\sigma^4 + \frac{97\,037}{2}\sigma^2 - 77\,328\right) = 0. \end{aligned} \tag{2.17}$$

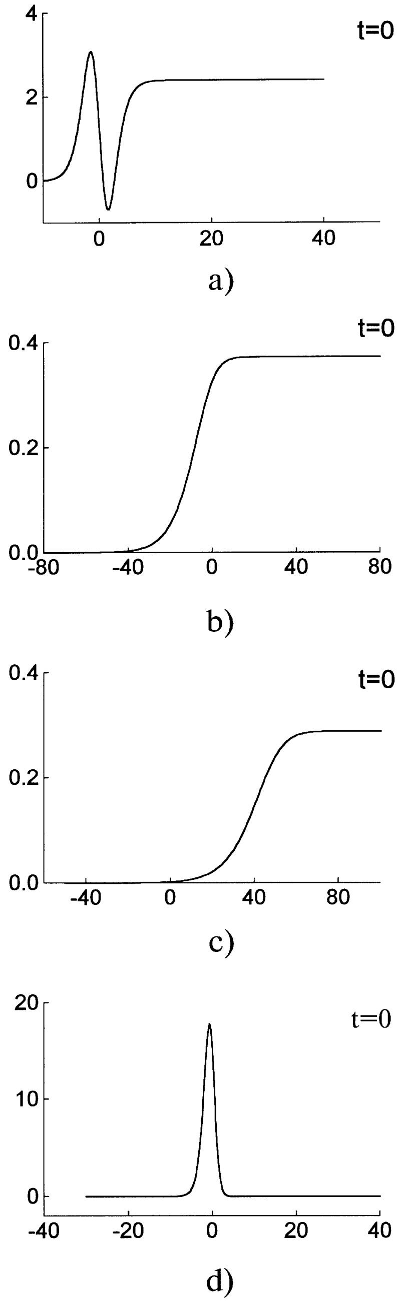

Figure 1. Special solutions of equation (1.6): (a) $\sigma = 0$, (b) $\sigma = 12/\sqrt{47}$, (c) $\sigma = 16/\sqrt{73}$ and (d) $\sigma = 4$.

Solving the latter set of equations results in C being an arbitrary constant, and σ and S which are presented in table 1.

Taking into account equations (2.13) and (2.14), one can find

$$\psi(x, t) = C_1\,\mathrm{e}^{k(x-Ct)/2} + C_2\,\mathrm{e}^{k(x-Ct)/2} \tag{2.18}$$

where $k^2 = -2S$, C_1 and C_2 are arbitrary constants.

Table 1. Solutions of equations (2.15)–(2.17).

σ	0	0	$\frac{12}{\sqrt{47}}$	$\frac{16}{\sqrt{73}}$	4	4
S	$-\frac{11}{38}$	$\frac{1}{38}$	$-\frac{1}{94}$	$-\frac{1}{146}$	$-\frac{1}{2}$	$\frac{1}{2}$

Reverting from function $\psi(x,t)$ taken in the form (2.18) to $Y = \psi_x/\psi$ brings about the following solution of equation (1.6) from transformation formula (2.5):

$$u = 120Y^3 - 15\sigma Y^2 + 15\left(4S + \frac{1}{76}(16-\sigma^2)\right)Y - 5\sigma S + \frac{\sigma}{76}\left(7 - \frac{13}{8}\sigma^2\right) + C \quad (2.19)$$

where

$$Y = \frac{k}{2}\tanh\left\{\frac{k}{2}(x-Ct) - \varphi_0\right\}$$

C and φ_0 are arbitrary constants. Indeed, expression (2.19) is the true solution of equation (1.6) with certain σ and S only as stated in table 1. In table 2 we list the solutions of equation (1.6) at concrete values of σ, $k = -\sqrt{2|S|}$, $\varphi_0 = 0$. Particular solutions of equation (1.6) corresponding to $\sigma = 0$, $12/\sqrt{47}$, $16/\sqrt{73}$ and 4 are plotted in figure 1. These are profiles of the solitary waves at $t = 0$. Clearly the wave propagation rate is determined by the choice of constant C. Here we take C so that $u \to 0$ at $\xi \to -\infty$, which gives $C = (30/11)k^3$, $60k^3$, $90k^3$ and $6k^3$, respectively, where k is from table 2.

Table 2. Special solutions of equation (1.6), $\xi = k/2(x - Ct)$.

σ	k	Special solutions
0	$\sqrt{\frac{11}{19}}$	$C + 15k^3\tanh(\xi)\left(\tanh^2(\xi) - \frac{9}{11}\right)$
0	$\frac{1}{\sqrt{19}}$	$C + 15k^3\tan(\xi)(\tan^2(\xi) + 3)$
$\frac{12}{\sqrt{47}}$	$\frac{1}{\sqrt{47}}$	$C + 15k^3\{[\tanh(\xi) - 1]^3 + 4\}$
$\frac{16}{\sqrt{73}}$	$\frac{1}{\sqrt{73}}$	$C + 15k^3\{\tanh(\xi)[\tanh^2(\xi) + 5] + 4\cosh^{-2}(\xi)\}$
4	1	$C + 4 - 15k^3[1 + \tan(\xi)]\cos^{-2}(\xi)$
4	1	$C - 6 + 15k^3[1 - \tanh(\xi)]\cosh^{-2}(\xi)$

Assuming $x \to \pm\infty$, we can obtain

$$u \to C + \frac{\sigma}{76}\left(7 - \frac{13}{8}\sigma^2\right) - \frac{5}{4}\sigma k^2 - \frac{15}{152}(16-\sigma^2)k \qquad \text{at } x \to -\infty \quad (2.20)$$

$$u \to C + \frac{\sigma}{76}\left(7 - \frac{13}{8}\sigma^2\right) - \frac{5}{4}\sigma k^2 + \frac{15}{152}(16-\sigma^2)k \qquad \text{at } x \to +\infty \quad (2.21)$$

where $k = \sqrt{-2S}$. Clearly, expressions (2.20) and (2.21) are true only for those S given in table 1 that are negative. Other solutions corresponding to $S > 0$ are periodical; moreover, they are singular at $\xi = \pm\pi/2$, and, as a result, cannot tend to any limit at $x \to \pm\infty$.

Taking an advantage of formulae (2.20) and (2.21) and solutions from table 2, one can build some approximate solutions of equation (1.1). Let us show, for example, that

$$u = 15k^3\{\tanh^3(\xi_+) + \tanh^3(\xi_-) - \tfrac{9}{11}(\tanh(\xi_+) + \tanh(\xi_-))\} \quad (2.22)$$

(where $\xi_\pm = (k/2)(x \mp \delta \mp Ct)$, $k = \sqrt{11}/\sqrt{19}$, $C = 30/11k^3$, $\delta \geqslant 5$) is the solution of equation (1.6). The approximate solution (2.22) was constructed from the solution of

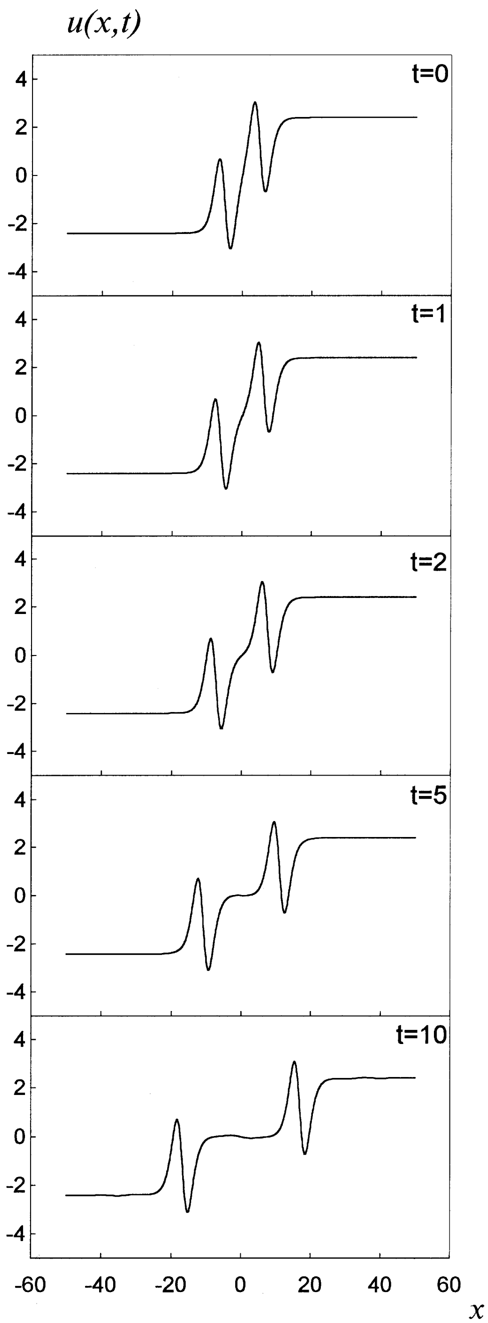

Figure 2. Approximate solution (2.2) of equation (1.6): $\sigma = 0$, $\delta = 5$.

equation (1.6) at $\sigma = 0$ given in table 2. It is equal to the sum of solutions u_1 and u_2, corresponding to $C_1 = C$ and $C_2 = -C$, and shifted by δ to the right and to the left along the x-axis accordingly. It is seen from figure 1 and expressions (2.20) and (2.21) that $u_1 \neq 0$ when $u_2 = 0$ and *vice versa*.

Solution (2.22) and the numerical solution of equation (1.6) more or less coincide for $\delta \geqslant 5$. Figure 2 illustrates the case $\delta = 5$.

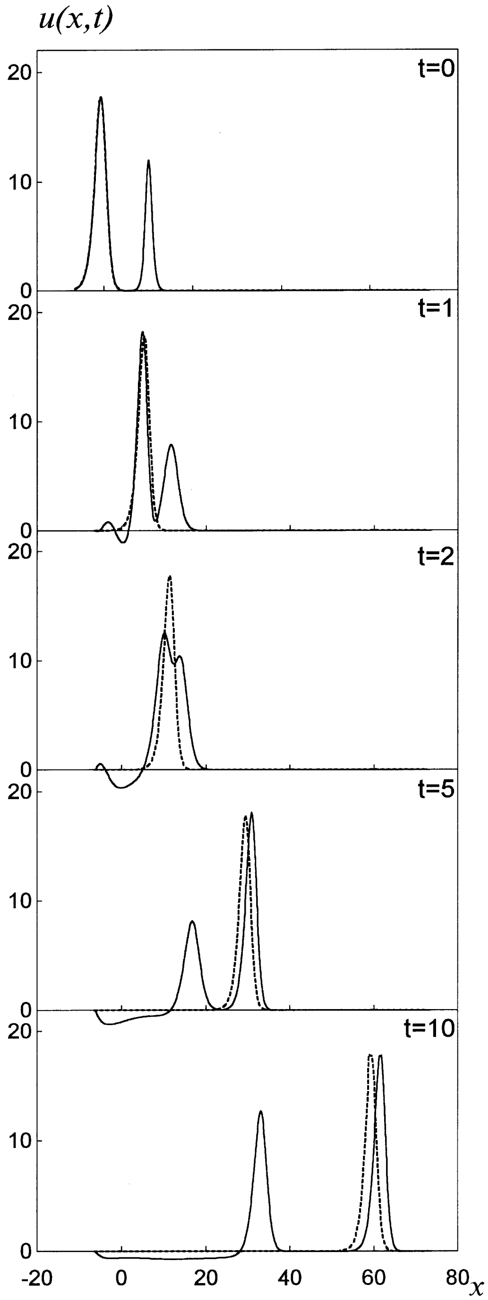

Figure 3. Interaction between the solitary wave (3.3) and the disturbance $u(x) = 15\cosh^{-2}\{8(x-3)\}$. The full curve shows the interaction; the broken curve is the analytical solution for the solitary wave only.

One can also present the solution of equation (1.6) in the form of a soliton lattice using the solution of equation (1.6) at $\sigma = 4$ from table 2. It takes the form

$$u \cong 15\sum_{n=1}^{N}[1-\tanh\{\tfrac{1}{2}(x-6t)+nT\}]\cosh^{-2}\{\tfrac{1}{2}(x-6t)+nT\} \qquad (2.23)$$

where T is the constant that satisfies the inequality $T \geqslant \frac{1}{6}$.

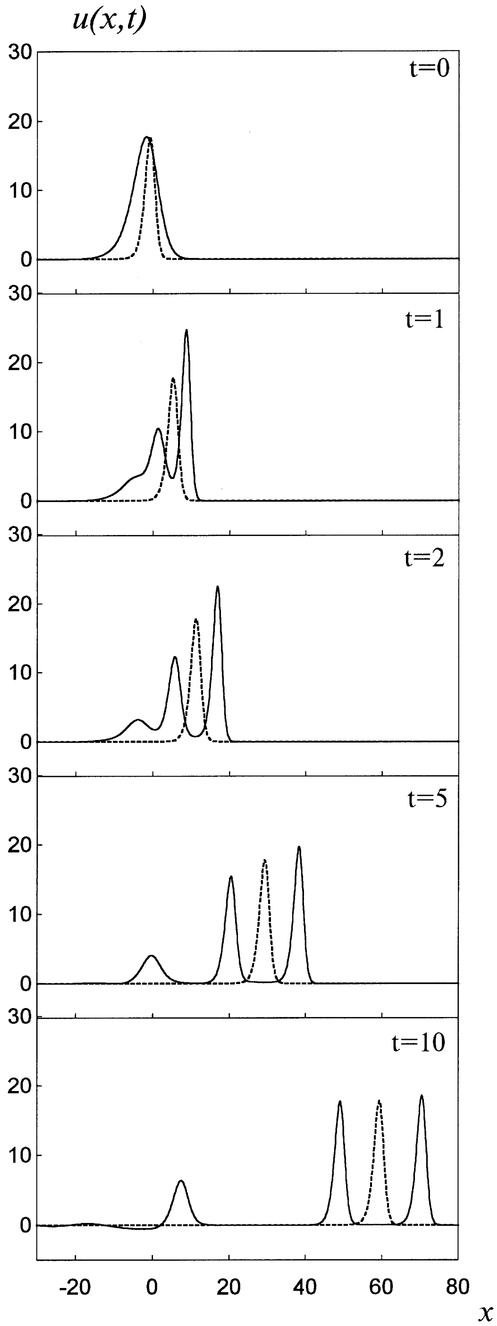

Figure 4. Propagation of the wave given at $t = 0$ by expression $u(x) = 15\cosh^{-2}(kx/2)\{1 - \tanh(kx/2)\}$, where $k = 0.4$ for the full curve, while $k = 1$ for the broken curve.

3. Numerical modelling solitary waves in active-dissipative dispersive media

So far we have been considering only a few special solutions for solitary waves. To investigate the propagation of waves and their interaction we shall use numerical simulation.

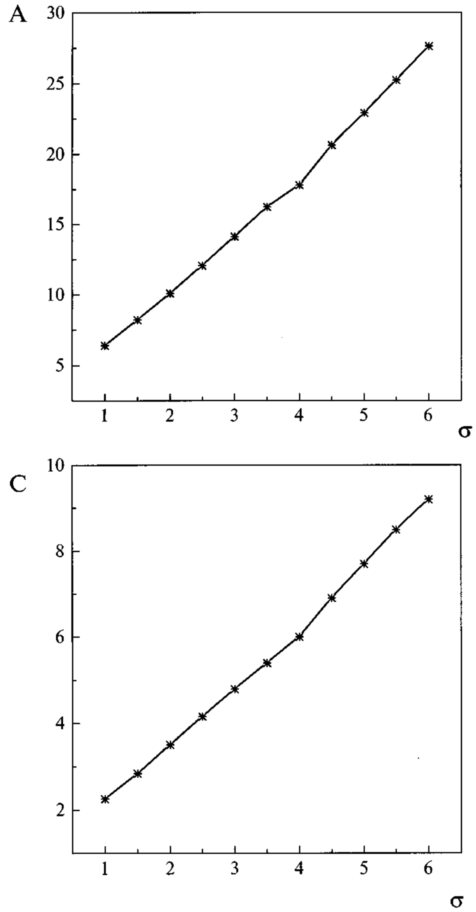

Figure 5. σ-dependence of amplitude A and velocity C of solitary waves formed from the initial condition (3.3).

In doing so, we employ the difference equation

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{1}{4h}[(u_{j+1}^n)^2-(u_{j-1}^n)^2]+\frac{\alpha}{2h^2}(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1})+\frac{\beta}{4h^3}(u_{j+2}^{n+1}-2u_{j+1}^{n+1}$$
$$+2u_{j-1}^{n+1}-u_{j-2}^{n+1})+\frac{\gamma}{2h^4}(u_{j+2}^{n+1}-4u_{j+1}^{n+1}+6u_j^{n+1}-4u_{j-1}^{n+1}+u_{j-2}^{n+1})=0 \qquad (3.1)$$

where u_j^{n+1} is the value of the mesh function at $x_j = jh$ and $t^{n+1} = (n+1)\tau$; τ is the mesh width in time; h is the mesh width along x-axis.

The difference equation (3.1) approximates the partial differential equation (1.1) with order $O(\tau + h^2)$. It can be transformed into the following equation,

$$a_j u_{j-2}^{n+1}+b_j u_{j-1}^{n+1}+c_j u_j^{n+1}+d_j u_{j+1}^{n+1}+e_j u_{j+2}^{n+1}=f_j \qquad (3.2)$$

where

$$a_j=-\frac{\beta h}{2}+\gamma \qquad b_j=\beta h-4\gamma+\alpha h^2$$
$$c_j=6\gamma-2\alpha h^2+\frac{2h^4}{\tau} \qquad d_j=-\beta h-4\gamma+\alpha h^2$$
$$e_j=\frac{\beta h}{2}+\gamma \qquad f_j=\frac{2h^4u_j^n}{\tau}-\frac{h^3}{2}[(u_{j+1}^n)^2-(u_{j-1}^n)^2].$$

The numerical scheme (3.1) is absolutely stable at $\tau \leqslant h^4/2\gamma$, because its realization leads us to the set of equations (3.2) with a band matrix. The difference equation (3.2) with appropriate initial condition $u(x, t = 0)$ and four boundary conditions added was programmed.

We took the special solutions of equation (1.1) from table 2 for testing convergence of the numerical solution obtained by difference equation (3.2) to the exact solution. We observed that an error accumulated in time. Apparently it was caused by using the linearized numerical scheme (3.1) when we took the nonlinear term from the previous time step. Keeping this in mind, we chose τ so that the numerical solution converged to the exact solution for rather large values of $t \sim 40$–50. The convergence proved to be driven by $\sigma = \beta/\sqrt{\alpha\gamma}$. So, at $\sigma = 0$ the difference scheme is applicable for $\tau \leqslant h^2$, whereas at $\sigma = 4$ the restrictions are more rigorous, $\tau \leqslant h^3$.

Numerical modelling the propagation of the solitary wave given at the initial instant by the expression

$$u(x) = 15\cosh^{-2}(x/2)\{1 - \tanh(x/2)\} \tag{3.3}$$

shows that at $\sigma = 4$ the solitary wave keeps its form and is transported with velocity $C = 6$. The profile of this wave when found numerically fits the solitary wave

$$u(x, t) = 15\cosh^{-2}((x - 6t)/2)\{1 - \tanh((x - 6t)/2)\} \tag{3.4}$$

which corresponds to (3.3) at $t = 0$.

We have also considered interactions between waves described by equation (1.1) and given at $t = 0$ by the formula

$$u(x, t = 0) = u_1 + u_2$$

where u_1 is (3.3) and $u_2 = 15\cosh^{-2}\{8(x - 3)\}$. Figure 3 illustrates the process of such an interaction. It appears from figure 3 that a wave corresponding to (3.4) at $t = 0$ encounters the disturbance, interacts with it, and continues unchanged, albeit with a certain phase lag, with respect to (3.4). Therefore the wave corresponding to (3.4) at $t = 0$ behaves like a soliton. However, the second wave alters with time, its amplitude increases, and for fairly large calculation times it acquires the same shape as solution (3.4). Note that this process of increasing amplitude takes place independently of the waves interaction.

We have studied the propagation of the solitary wave in active-dissipative dispersive media at another initial condition. Figure 4 illustrates the evolution of the solitary wave that is wider at $t = 0$ than one described by (3.3):

$$u(x) = 15\cosh^{-2}(0.4(x/2))\{1 - \tanh(0.4(x/2))\}.$$

One can see that this wave falls into pieces with time, resulting in three waves. Two of them have the same shape as the exact solitary wave solution (3.4). The broken curve in figure 4 illustrates the exact solution (3.4) of equation (1.1). The initial wave with larger amplitude than that of the exact solution (3.4) is also split into pieces and we obtain a solution like (3.4). In contrast, if the amplitude of the initial wave is smaller than that of solution (3.4), this wave will grow with time until it achieves the amplitude of the analytical solution.

It should be noted that such results can be observed at different values of σ. In examining the effect of the parameter σ on the structure formation of a solitary wave we found a numerical solution of equation (1.1) at different values of σ with (3.3) as an initial condition. We observed this wave developing in time and acquired the form of the solitary wave with the amplitude and the wavenumber corresponding to each specific σ. However, at $\sigma < 0.5$ we noticed no regular structure to be organized. The σ-dependence of the amplitude and speed of the solitary wave formed from the initial condition (3.3) is plotted in figure 5. This figure illustrates that both the amplitude and wave speed of the soliton solution increase with σ.

4. Conclusion

Solitary waves in active-dissipative dispersive media were considered. A procedure of finding solitary wave solutions of equation (1.1) by means of the singular manifold method was given. All known localized solutions of equation (1.1) were listed in table 2. The difference scheme was offered to simulate the interaction of solitary waves and study solutions of equation (1.1) in the general case of parameters α, β and γ. The σ-dependence of the amplitude and speed of the solitary wave, formed from the localized disturbance, was obtained.

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References

[1] Topper J and Kawahara T 1978 *J. Phys. Soc. Japan* **44** 663
[2] Tatsumi T 1984 *Turbulence and Chaotic Phenomena of Fluids* (Iutam)
[3] Nicolaevsky V N 1985 *Dokl. Akad. Nauk SSSR* **283** 1321
[4] Kuramoto Y and Tsuzuki T 1976 *Prog. Theor. Phys.* **55** 356
[5] Sivashinsky G I 1983 *Ann. Rev. Fluid Mech.* **15** 179
[6] Kawahara T 1983 *Phys. Rev. Lett.* **51** 381
[7] Conte R and Musette M 1989 *J. Phys. A: Math. Gen.* **22** 169
[8] Kudryashov N A 1988 *Prikl. Mat. Mekh.* **52** 465 (1988 *J. Appl. Math. Mech.* **52** 361)
[9] Kudryashov N A 1990 *Phys. Lett.* **147A** 287
[10] Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
[11] Conte R 1989 *Phys. Lett.* **140A** 383
[12] Conte R 1989 *Phys. Lett.* **134A** 100